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DESIGNING VIBRATING BEAMS AND ROTATING SHAFTS FOR MAXIMUM DIFFERENCE BETWEEN ADJACENT NATURAL FREQUENCIES

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Abstract—Using the cross-sectional area function as the design variable, we determine the optimal design of a transversely vibrating, thin, elastic beam or rotating shaft that maximizes the difference $\omega_n - \omega_{n-1}$ between two adjacent natural frequencies or critical whirling speeds ω_n and ω_{n-1} of given order. The beams have geometrically similar cross-sections while the shafts are restricted to be circular, and a minimum constraint is prescribed for the cross-sectional area. The volume, length, and boundary conditions are assumed to be given, and the beams or shafts may be equipped with given, non-structural masses or disks.

The set of governing equations, which is the same for the two different types of physical problem under consideration, are derived by variational analysis and solved numerically by a successive iteration procedure based on a finite difference discretization. A number of optimal solutions for cantilevers are presented in the paper, and comparisons are made with results obtained earlier for the similar, but simpler problem of maximizing the value of a single, higher order natural frequency or whirling speed, ω_n.

1. INTRODUCTION

The vast majority of problems of optimal design against structural vibration and instability dealt with in the literature consist in maximizing a fundamental eigenvalue for given structural volume, or in minimizing the volume for prescribed fundamental eigenvalue. The reader is referred to recent review papers[1–6]. The practical significance of optimizing with respect to a fundamental eigenvalue, i.e. the fundamental natural vibration frequency of a bar, beam or plate, or the first critical whirling speed of a rotating shaft, is that one obtains a design of minimum weight (or cost) against resonance or whirling instability, respectively, subject to all external vibration frequencies or service speeds within a large range from zero and up to the particular fundamental eigenvalue.

However, for problems of resonance due to external excitation frequencies, or whirling instability at service speeds, where the external frequencies or the service speeds are confined within a given range of *finite* upper and lower limits, much more competitive designs may be obtained by maximizing the distance between two adjacent natural frequencies or critical speeds; this being done in such a way that the excitation frequency range lies between the two natural frequencies in question.

This type of problem has previously been considered by Troitskii in Ref.[7], where a mathematical formulation is given (but no solutions presented) for the problem of maximizing the difference $\omega_2^2 - \omega_1^2$ between the squared second and first natural frequencies for axial vibrations of a bar of given volume and length. The optimality condition for this problem is slightly different from the optimality condition associated with the more realistic problem of maximizing the frequency difference $\omega_2 - \omega_1$, and may only be expected to lead to the same optimal solution, if, as in[7], no constraints are specified for the cross-sectional area of the bar.

In the present paper, we consider problems of determining the distribution of structural material of transversely vibrating beams or of rotating circular shafts, such that maximum difference $\omega_n - \omega_{n-1}$ is obtained between two adjacent frequencies or critical whirling speeds ω_n and ω_{n-1} of given orders n and $n-1$. The volume, length, and boundary conditions are assumed to be given for the beam or shaft, which may be equipped with a given set of non-structural masses or disks. A geometric constraint, namely a minimum allowable value for the cross-sectional area, is used in our formulation for optimal design.

If the cross-sectional area function is unconstrained (except for the given volume), then the solutions to our optimization problem are the same as the solutions to the different problem of

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maximizing a single, higher order natural frequency ω_n of given order n, for specified volume and length of the structure. The latter problem is treated in Ref. [8], where a number of optimal designs are available. The reason why the two different optimization problems have identical solutions, is the following: when a single, higher order natural frequency ω_n is maximized without specification of a geometric minimum constraint, *all* the natural frequencies of orders lower than n become associated with rigid body motions and attain zero value, see [8]. Obviously, maximum ω_n implies maximum difference $\omega_n - \omega_{n-1}$ under such conditions. In that problem, singularities of zero cross-sectional area occur, giving rise to inner separations and hinges.

In Ref. [9], single, higher order natural frequencies ω_n are maximized while a geometric minimum constraint is taken into account, and the lower order natural frequencies then all become finite. It was conjectured in Ref. [9] that the frequency differences found, are close to the maximum obtainable. This conjecture is confirmed by way of examples in the present study, which deals with the more complicated problem of direct maximization of the difference between adjacent frequencies. Here, two eigenfunctions appear nonlinearly in the optimality condition.

2. FREE TRANSVERSE VIBRATION OF BEAMS AND WHIRLING OF SHAFTS

We consider a Bernoulli–Euler beam or shaft of length L and structural volume V, made of a linearly elastic material with Young's modulus E and the mass density ρ. The structure has variable, but geometrically similar cross-sections (in the case of a shaft, we assume circular cross-sections) with the relationship $I = cA^2$ between the moment of inertia I and the area A, where the positive constant c is given by the cross-sectional geometry. A number, M, of given non-structural masses Q_i, $i = 1, \ldots, M$, (circular disks in the case of a shaft) are assumed to be attached to the structure at prescribed points $X = X_i$, $i = 1, \ldots, M$, along the coordinate axis, X.

Denoting by Ω_j the jth natural angular frequency of transverse vibrations or the jth critical angular whirling speed if the structure is a beam or a rotating shaft, respectively, we now introduce the following dimensionless quantities: coordinate x, cross-sectional area $\alpha(x)$, non-structural masses q_i, and natural frequencies (or critical speeds) ω_j, where

$$\begin{aligned} x &= X/L, 0 \leq x \leq 1, \\ \alpha(x) &= A(x)\frac{L}{V}, \\ q_i &= \frac{Q_i}{\rho V}, x = x_i, i = 1, \ldots, M, \\ \omega_j^2 &= \frac{\Omega_j^2 \rho L^5}{cEV}. \end{aligned} \tag{1}$$

The eigenfunction (i.e. vibration or whirling mode) y_j associated with the jth natural frequency or critical speed ω_j of the transversely vibrating beam or rotating shaft, is governed by the dimensionless differential equation

$$(\alpha^2 y_j'')'' = \omega_j^2 \alpha y_j, \tag{2}$$

in which rotational inertia in the case of a beam and gyroscopic effects for a shaft are neglected. In addition to (2), appropriate boundary conditions must be specified. We assume, for simplicity, that the structure has clamped, simply supported or free ends, i.e. that no flexible or intermediate supports are present. If the cross-sectional area is everywhere larger than zero, the eigenfunction y_j, its first derivative y_j' and the bending moment $\alpha^2 y_j''$ are continuous throughout, but the shear force $(\alpha^2 y_j'')'$ has discontinuities (jumps) given by

$$\langle(\alpha^2 y_j'')'\rangle_{x_i} = (\alpha^2 y_j'')'_{x=x_i^+} - (\alpha^2 y_j'')'_{x=x_i^-} = \omega_j^2 q_i y_j(x_i), \quad i = 1, \ldots, M, \tag{3}$$

at the points $x = x_i$, $i = 1, \ldots, M$, where the masses q_i are attached to the structure.

Multiplying (2) by y_j, integrating twice by parts over the interval $0 \le x \le 1$ and taking (3) and the boundary conditions into account, we obtain the following Rayleigh expression for the eigenvalue ω_j^2,

$$\omega_j^2 = \frac{\int_0^1 \alpha^2 y_j''^2 \, dx}{\int_0^1 \alpha y_j^2 \, dx + \sum_{i=1}^{M} q_i y_j^2(x_i)}. \tag{4}$$

If ω_j^2 is a higher order eigenvalue ($j > 1$), then Rayleigh's principle states that the right hand side of (4), i.e. the Rayleigh quotient, is stationary at the eigenvalue ω_j^2 corresponding to the actual eigenfunction y_j among all kinematically admissible deflection functions that are orthogonal to all the lower eigenfunctions y_k, $k = 1, \ldots, j-1$. Together with a normalization condition for y_j, which makes the denominator of the Rayleigh quotient in (4) equal to unity, these orthogonality conditions can be written as

$$\int_0^1 \alpha y_j y_k \, dx + \sum_{i=1}^{M} q_i y_j(x_i) y_k(x_i) = \delta_{jk}, \quad k = 1, \ldots, j, \tag{5}$$

where δ_{jk} denotes Kronecker's delta.

We consider the problem of finding the distribution of cross-sectional area, $\alpha(x)$, of a beam or rotating shaft of given length and volume, for which the difference between two adjacent higher order natural frequencies ω_n and ω_{n-1} is to be maximized for given $n > 1$. We shall assume that both frequencies are single modal and that $\omega_{n-1} > 0$. Noting that in nondimensional form the length of the beam or shaft is unity, we may express the volume constraint after nondimensionalization by

$$\int_0^1 \alpha \, dx = 1. \tag{6}$$

In addition, we consider a geometric minimum constraint, $\alpha(x) \ge \bar{\alpha}$, to be specified for the cross-sectional area function $\alpha(x)$, where the minimum allowable value $\bar{\alpha}$, $0 < \bar{\alpha} < 1$, is assumed to be given. By introduction of the real slack variable $g(x)$, the constraint $\alpha(x) \ge \bar{\alpha}$ can be expressed via the equality constraint

$$g^2(x) = \alpha(x) - \bar{\alpha}. \tag{7}$$

3. VARIATIONAL ANALYSIS

A variational formulation of the optimization problem stated above is employed, using the functional

$$\begin{aligned} F = {} & \left[\int_0^1 \alpha^2 y_n''^2 \, dx\right]^{1/2} - \left[\int_0^1 \alpha^2 y_{n-1}''^2 \, dx\right]^{1/2} \\ & - \sum_{k=1}^{n} \Lambda_k \left[\int_0^1 \alpha y_n y_k \, dx + \sum_{i=1}^{M} q_i y_n(x_i) y_k(x_i) - \delta_{nk}\right] \\ & - \sum_{k=1}^{n-1} \lambda_k \left[\int_0^1 \alpha y_{n-1} y_k \, dx + \sum_{i=1}^{M} q_i y_{n-1}(x_i) y_k(x_i) - \delta_{(n-1)k}\right] \\ & - \sum_{k=1}^{n-2} \int_0^1 \mu_k(x) [(\alpha^2 y_k'')'' - \omega_k^2 \alpha y_k] \, dx \\ & - \beta\left[\int_0^1 \alpha \, dx - 1\right] - \int_0^1 \kappa(x)[g^2(x) - \alpha(x) + \bar{\alpha}] \, dx, \end{aligned} \tag{8}$$

where the frequency difference $\omega_n - \omega_{n-1}$ is expressed as a functional via (4) with unit value of the denominator, and is augmented by the orthonormality conditions (5) for the mode y_n, similar conditions for the mode y_{n-1}, the differential constraints (2) for the lower order modes y_k, $k = 1, \ldots, n-2$, the volume constraint (6), and the geometric minimum constraint (7). These side conditions are introduced by means of the Lagrangian multipliers Λ_k $(k = 1,\ldots,n)$, λ_k $(k = 1,\ldots,n-1)$, $\mu_k(x)$ $(k = 1,\ldots,n-2)$, β, and $\kappa(x)$, respectively. In view of Rayleigh's principle for higher order eigenvalues and our introduction of the Lagrangian multipliers, we may take the variations of F with respect to y_k $(k = 1,\ldots,n)$, ω_k^2 $(k = 1,\ldots,n-2)$, α and g independently in the following.

The condition of stationarity of F with respect to variation δy_k, $k = 1,\ldots,n$, becomes

$$\begin{aligned}
&(\omega_n^{-1}\delta_{nk} - \omega_{n-1}^{-1}\delta_{(n-1)k})\left[[\alpha^2 y_k''\delta y_k']_0^1 - [(\alpha^2 y_k'')'\delta y_k]_0^1 + \sum_{i=1}^{M}\langle(\alpha^2 y_k'')'\rangle_{x_i}\delta y_k(x_i) + \int_0^1 (\alpha^2 y_k'')''\delta y_k\,dx\right] \\
&- \Lambda_k(1+\delta_{nk})\left[\int_0^1 \alpha y_n \delta y_k\,dx + \sum_{i=1}^{M} q_i y_n(x_i)\delta y_k(x_i)\right] - \lambda_k(1+\delta_{(n-1)k})(1-\delta_{nk})\left[\int_0^1 \alpha y_{n-1}\delta y_k\,dx\right. \\
&\left.+\sum_{i=1}^{M} q_i y_{n-1}(x_i)\delta y_k(x_i)\right] - \delta_{nk}\sum_{p=1}^{n-1}\Lambda_p\left[\int_0^1 \alpha y_p \delta y_k\,dx + \sum_{i=1}^{M} q_i y_p(x_i)\delta y_k(x_i)\right] - \delta_{(n-1)k}\sum_{p=1}^{n-2}\lambda_p\left[\int_0^1 \alpha y_p\delta y_k\right. \\
&\qquad\left.+\sum_{i=1}^{M} q_i y_p(x_i)\delta y_k(x_i)\right] - e_k\left[[(\alpha^2\delta y_k'')'\mu_k]_0^1 - [\alpha^2\delta y_k''\mu_k']_0^1 + [\alpha^2\mu_k''\delta y_k']_0^1 - [(\alpha^2\mu_k'')'\delta y_k]_0^1\right. \\
&\qquad+\sum_{i=1}^{M}\langle\mu_k'\rangle_{x_i}\alpha^2\delta y_k''(x_i) - \sum_{i=1}^{M}\langle\alpha^2\mu_k''\rangle_{x_i}\delta y_k'(x_i) + \sum_{i=1}^{M}[\langle(\alpha^2\mu_k'')'\rangle_{x_i} - \omega_k^2 q_i\mu_k(x_i)]\delta y_k(x_i) \\
&\qquad\qquad\left.+\int_0^1 [(\alpha^2\mu_k'')'' - \omega_k^2\alpha\mu_k]\delta y_k\,dx\right] = 0, \quad k = 1,\ldots,n, \qquad (9)
\end{aligned}$$

after integration by parts, using the continuity conditions and the normalization (5), (with $j = k$) for the modes y_k, $k = 1,\ldots,n$. The jump conditions (3) are also used for $k = 1,\ldots,n-2$, and (4) is employed for $k = j = n-1$ and $k = j = n$. The symbol e_k introduced in (9), is defined by

$$e_k = \begin{cases} 1 \text{ for } k = 1,\ldots,n-2 \\ 0 \text{ for } k = n-1, n \end{cases}. \qquad (10)$$

Now, since the variations $\delta y_k(x)$, $k = 1,\ldots,n$, are arbitrary, it follows from (9) that the differential equations

$$\begin{aligned}
&(\omega_n^{-1}\delta_{nk} - \omega_{n-1}^{-1}\delta_{(n-1)k})(\alpha^2 y_k'')'' - e_k[(\alpha^2\mu_k'')'' - \omega_k^2\alpha\mu_k] \\
&= \Lambda_k(1+\delta_{nk})\alpha y_n + \lambda_k(1+\delta_{(n-1)k})(1-\delta_{nk})\alpha y_{n-1}, \\
&+ \delta_{nk}\sum_{p=1}^{n-1}\Lambda_p\alpha y_p + \delta_{(n-1)k}\sum_{p=1}^{n-2}\lambda_p\alpha y_p, \quad k = 1,\ldots,n, \qquad (11)
\end{aligned}$$

must hold in the interval $0 \le x \le 1$, and that the following conditions must hold at the points $x = x_i$, $i = 1,\ldots,M$:

$$\begin{aligned}
&(\omega_n^{-1}\delta_{nk} - \omega_{n-1}^{-1}\delta_{(n-1)k})\langle(\alpha^2 y_k'')'\rangle_{x_i} - e_k[\langle(\alpha^2\mu_k'')'\rangle_{x_i} - \omega_k^2 q_i\mu_k(x_i)] \\
&= \Lambda_k(1+\delta_{nk})q_i y_n(x_i) + \lambda_k(1+\delta_{(n-1)k})(1-\delta_{nk})q_i y_{n-1}(x_i), \quad i = 1,\ldots,M, \quad k = 1,\ldots,n. \\
&+ \delta_{nk}\sum_{p=1}^{n-1}\Lambda_p q_i y_p(x_i) + \delta_{(n-1)k}\sum_{p=1}^{n-2}\lambda_p q_i y_p(x_i), \qquad (12)
\end{aligned}$$

At the latter points, (9) also yields the continuity conditions

$$\left.\begin{aligned}\langle\alpha^2\mu_k''\rangle_{x_i} = 0 \\ \langle\mu_k'\rangle_{x_i} = 0\end{aligned}\right\} i = 1,\ldots,M, \quad k = 1,\ldots,n-2, \qquad (13)$$

for the Lagrangian multipliers $\mu_k(x)$, because the variations $\delta y_k'(x_i)$ and $\alpha^2\delta y_k''(x_i)$, $i = 1,\ldots,M$, $k = 1,\ldots,n-2$, are arbitrary.

In addition to (11)–(13), equation (9) yields, for $k = n$ and $k = n-1$, the natural boundary conditions that $(\alpha^2 y_k'')'$ and $\alpha^2 y_k''$ must vanish at the ends of the structure, if y_k and y_k', respectively, are not specified there. The natural boundary conditions furnished by (9) for

$k = 1, \ldots, n-2$, are easily seen to imply that the Lagrangian multipliers $\mu_k(x)$ must satisfy the same boundary conditions as the lower order modes y_k, $k = 1, \ldots, n-2$.

In order to identify the Lagrangian multipliers Λ_k $(k = 1, \ldots, n)$, λ_k $(k = 1, \ldots, n-1)$ and $\mu_k(x)$ $(k = 1, \ldots, n-2)$, we first multiply (11) by $y_j(x)$, $j = 1, \ldots, n$, and integrate twice by parts over the interval $0 \leq x \leq 1$, using the boundary conditions and the conditions of continuity of y_j and y'_j. Then after making use of (12), (13) and (5), we obtain the equation

$$(\omega_n^{-1}\delta_{nk} - \omega_{n-1}^{-1}\delta_{(n-1)k}) \int_0^1 \alpha^2 y_k'' y_j'' \, dx - \epsilon_k \left[\int_0^1 \alpha^2 u_k'' y_j'' \, dx - \omega_k^2 \left[\int_0^1 \alpha\mu_k y_j \, dx + \sum_{i=1}^{M} q_i \mu_k(x_i) y_j(x_i)\right]\right]$$
$$= \Lambda_k \delta_{nj}(1 + \delta_{nk}) + \lambda_k \delta_{(n-1)j}(1 + \delta_{(n-1)k})(1 - \delta_{nk}), + \Lambda_j \delta_{nk}(1 - \delta_{nj}) + \lambda_j \delta_{(n-1)k}(1 - \delta_{nj})(1 - \delta_{(n-1)j}),$$
$$k = 1, \ldots, n, \qquad j = 1, \ldots, n. \tag{14}$$

Substituting $k = j = n$; $k = n-1$, $j = n$; $k = n$, $j = n-1$; and $k = j = n-1$, respectively, into (14), we find, after use of (4), (5) and (10),

$$\Lambda_n = \frac{1}{2}\omega_n, \quad \Lambda_{n-1} = -\omega_{n-1}^{-1} \int_0^1 \alpha^2 y_{n-1}'' y_n'' \, dx = 0,$$
$$\lambda_{n-1} = -\frac{1}{2}\omega_{n-1}. \tag{15}$$

We now multiply by y_n and y_{n-1}, respectively, the differential equations (2) for $y_j(x)$, $j = 1, \ldots, n-2$, which are contained as constraints in the functional (8). Integrating twice by parts and making use of the jump conditions (3) for y_j, we obtain $\int_0^1 \alpha^2 y_n'' y_j'' \, dx = 0$ and $\int_0^1 \alpha^2 y_{n-1}'' y_j'' \, dx = 0$, $j = 1, \ldots, n-2$. If we consider (14) for $j = 1, \ldots, n-2$, and first substitute $k = n$ and then $k = n-1$, we find that $\omega_n \Lambda_j$ and $\omega_{n-1}\lambda_k$, respectively, are equal to the integrals just written. Hence, we have

$$\Lambda_j = \omega_n^{-1} \int_0^1 \alpha^2 y_n'' y_j'' \, dx = 0, \quad \lambda_j = \omega_{n-1}^{-1} \int_0^1 \alpha^2 y_{n-1}'' y_j'' \, dx = 0,$$
$$j = 1, \ldots, n-2. \tag{16}$$

If we substitute the results (15) and (16) into (11) and (12) with $k = n$ and $k = n-1$, we establish the differential equation (2) and the jump condition (3) for the modes $y_n(x)$ and $y_{n-1}(x)$.

Furthermore, substituting Eqs. (16) into (11) and (12) with $k = 1, \ldots, n-2$, we find that the Lagrangian multipliers $\mu_k(x)$, $k = 1, \ldots, n-2$, satisfy the same differential equations and jump conditions as those governing the functions $y_k(x)$, $k = 1, \ldots, n-2$, i.e. Eqs. (2) and (3). Since $\mu_k(x)$ have also been found to satisfy the same boundary and continuity conditions as the modes y_k, $k = 1, \ldots, n-2$, we have $\mu_k(x) = c_k y_k(x)$, $k = 1, \ldots, n-2$, where c_k are constants.

Now, the condition of stationarity of the functional F in (8) with respect to any of the eigenvalues ω_k^2, $k = 1, \ldots, n-2$, gives

$$\int_0^1 \alpha\mu_k y_k \, dx = 0, \quad k = 1, \ldots, n-2. \tag{17}$$

Substituting into (17) the relationships $\mu_k = c_k y_k$ just found, we obtain $c_k = 0$ since $\int_0^1 \alpha y_k^2 \, dx > 0$, $k = 1, \ldots, n-2$. This implies that all the Lagrangian multipliers $\mu_k(x)$ vanish in the interval $0 \leq x \leq 1$,

$$\mu_k(x) = 0, \quad k = 1, \ldots, n-2. \tag{18}$$

The condition of stationarity of the functional F with respect to variation $\delta\alpha$ of the cross-sectional area function is exp-essed by

$$\left[\int_0^1 \alpha^2 y_n''^2 \, dx\right]^{-1/2} \int_0^1 \alpha y_n''^2 \delta\alpha \, dx - \left[\int_0^1 \alpha^2 y_{n-1}''^2\right]^{-1/2} \int_0^1 \alpha y_{n-1}''^2 \delta\alpha \, dx$$
$$- \sum_{k=1}^{n} \Lambda_k \int_0^1 y_n y_k \delta\alpha \, dx - \sum_{k=1}^{n-1} \lambda_k \int_0^1 y_{n-1} y_k \delta\alpha \, dx - \beta \int_0^1 \delta\alpha \, dx + \int_0^1 \kappa(x)\delta\alpha \, dx = 0, \tag{19}$$

where Eqs. (18) have been taken into account. Collecting terms, using (4) and (5) for $j = k = n$ and $j = k = n - 1$, using (15) and (16), and recalling that the variation $\delta\alpha$ is arbitrary, we obtain the equation

$$\alpha[\omega_n^{-1}y_{n-1}''^2 - \omega_{n-1}^{-1}y_{n-1}''^2] - \frac{1}{2}[\omega_n y_n^2 - \omega_{n-1}y_{n-1}^2] = \beta - \kappa(x), \tag{20}$$

which constitutes the so-called optimality condition for our problem. If we introduce the dimensionless bending moment functions

$$m_n(x) = \alpha^2 y_n'', \quad m_{n-1}(x) = \alpha^2 y_{n-1}'', \tag{21}$$

and eliminate the second derivatives y_n'' and y_{n-1}'' in (20), we get

$$\alpha^{-3}[\omega_n^{-1}m_n^2 - \omega_{n-1}^{-1}m_{n-1}^2] - \frac{1}{2}[\omega_n y_n^2 - \omega_{n-1}y_{n-1}^2] = \beta - \kappa(x). \tag{22}$$

The stationarity of the functional F in (8) with respect to arbitrary variation of the slack variable $g(x)$ leads to the switching equation

$$\kappa(x)g(x) = 0. \tag{23}$$

This equation implies that if $g \neq 0$ at a given value of x, then $\kappa = 0$, which simplifies (20) and (22), and by (7), we have $\alpha > \bar{\alpha}$. Thus, the optimal cross-sectional area function α is unconstrained at such a value of x, and we may write $x \in x_u$, where x_u designates the union(s) of unconstrained sub-intervals in the optimal solution. If, on the other hand, $g = 0$ at a given x, then $\alpha = \bar{\alpha}$ by (7), and the optimal cross-sectional area function is constrained. The corresponding value of x then belongs to the union(s) of constrained sub-intervals, which we will denote by x_c. We prefer to use x_u and x_c rather then $\kappa(x)$ and $g(x)$ in the formulation of our problem, and may now write from (22) the following equations for the optimal cross-sectional area function $\alpha(x)$:

$$\alpha^{-3}[\omega_n^{-1} m_n^2 - \omega_{n-1}^{-1}m_{n-1}^2] - \frac{1}{2}[\omega_n y_n^2 - \omega_{n-1}y_{n-1}^2] = \beta \ \text{(if } \alpha > \bar{\alpha}), \ x \in x_u; \quad \alpha = \bar{\alpha}, \ x \in x_c. \tag{24}$$

4. METHOD OF SOLUTION

For given mode order n, minimum allowable cross-sectional area $\bar{\alpha}$ and set of attached masses q_i, $i = 1, \ldots, M$, we are required to find the eigenfunctions y_j and eigenfrequencies ω_j, $j = 1, \ldots, n$, the Lagrangian multiplier β, and the optimal cross-sectional area function $\alpha(x)$ together with the constrained and unconstrained regions x_c and x_u.

The solution must satisfy the optimality condition (24), the volume constraint (6), the differential equations (2) for all modes, the "jump" conditions (3), the Rayleigh expression (4) and the orthonormality conditions (5), as well as suitable boundary conditions. The examples reported here deal with a cantilever beam or shaft for which the deflection and slope are zero at the fixed end and the moment and shear force are zero at the free end, i.e.

$$\left.\begin{aligned} y_j(0) &= 0 \\ y_j'(0) &= 0 \\ \alpha^2 y_j''(1) &= 0 \\ (\alpha^2 y_j'')'_{x=1} &= 0. \end{aligned}\right\} \quad j = 1, \ldots, n \tag{25}$$

In order to put the optimality condition in the form used in the computational algorithm we consider (22) in the region $x \in x_u$, in which $\kappa(x) = 0$. Dividing each side by β, raising each side to a power r and multiplying by α, we obtain

$$\alpha_{i+1}(x)=\begin{cases}\alpha_i\Big[\beta^{-1}\Big\{\alpha_i^{-3}[(\omega_n)_i^{-1}(m_n)_i^2-(\omega_{n-1})_i^{-1}(m_{n-1})_i^2] \\ \qquad -\frac{1}{2}[(\omega_n)_i(y_n)_i^2-(\omega_{n-1})_i(y_{n-1})_i^2]\Big\}\Big]^r \text{ (if} > \bar{\alpha}), x\in(x_u)_{i+1}; \\ \bar{\alpha},\ x\in(x_c)_{i+1}.\end{cases} \tag{26}$$

Here, the subscript i refers to the ith iteration.

The solution is based on a discrete representation of the variables at a large number of equally spaced points in the interval $0\le x\le 1$. The integrations and other computations are performed numerically.

The computation scheme is as follows:

(i) Assume starting values for $\alpha(x)$.

(ii) Determine the set of orthonormal eigenfunctions y_j, their derivatives y_j', y_j'', and associated eigenvalues ω_j, $j=1,\ldots,n$, from the eqs (2) to (5) and (25).

(iii) Calculate new values of $\alpha(x)$, and the sub-intervals x_u and x_c, from (26), adjusting the value of β so that (6) is satisfied.

(iv) Repeat from (ii) until the iterates become stationary.

The procedure is started by assuming $\alpha(x)$, and in many of the examples calculated α was

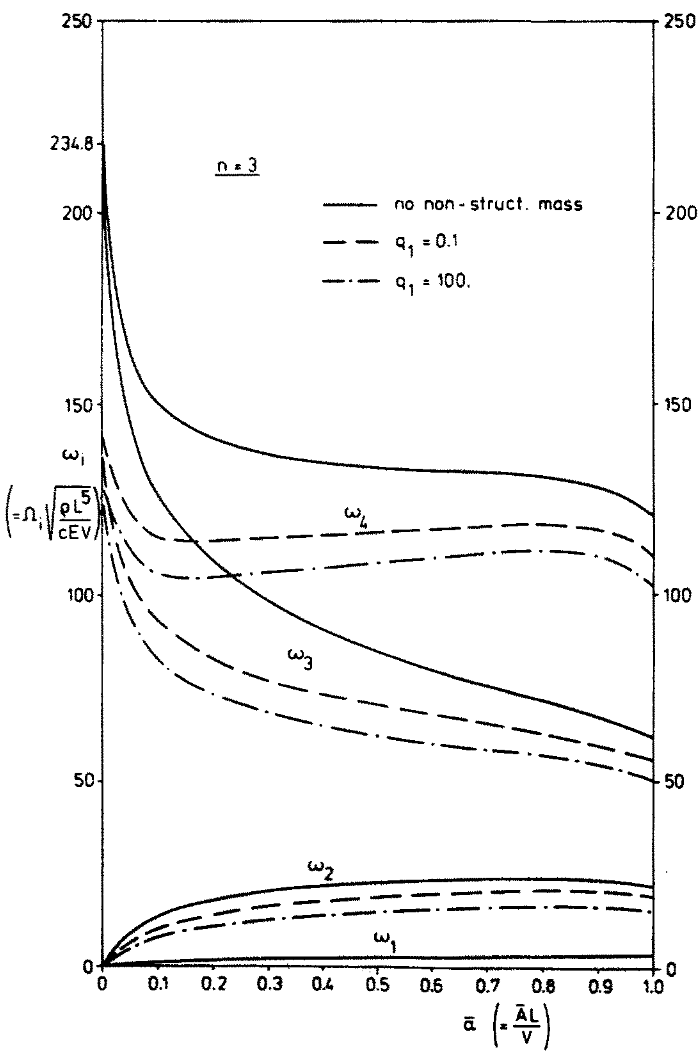

Fig. 1. Eigenfrequencies of optimal $\omega_3-\omega_2$ cantilevers vs. minimum cross-sectional area constraint $\bar{\alpha}$.

simply set equal to unity throughout the interval $0 \leq x \leq 1$. However, when a series of examples was to be calculated, with the same value of n and the same set of non-structural masses, it was usually economical in terms of computer time to use one of solutions already found as a starting point.

A common procedure for computing eigenfunctions and eigenvalues associated with a given cross-sectional area function is used in step (ii). The orthonormalization is performed by the well-known Gram process.

In step (iii) the value of β is found by a simple step search.

The power r in eq (26) must be chosen to give a balance between stability and speed of convergence. The most suitable value appears to change with changes in $\bar{\alpha}$, the attached masses q_i and the number of intervals used in the calculation. Values of r between 0.05 and 0.2 were used in the examples reported here.

The convergence was rather slow and in most cases it was necessary to apply a very sharp convergence criterion to ensure that the process was not stopped before the solution was reached.

In some cases the iteration converged to a stationary solution which was not the global optimum. This could be avoided to a large degree by changing the value of r and/or using a different starting approximation.

5. RESULTS

We present the results for two sets of examples, namely, $n = 3$ with a single non-structural mass at the free end $x = 1$, and $n = 5$ with two equal masses at $x = \frac{1}{2}$ and at $x = 1$. In each case solutions were found for values of mass q from 0 to 100, and for values of $\bar{\alpha}$ from 0.1 to 1.

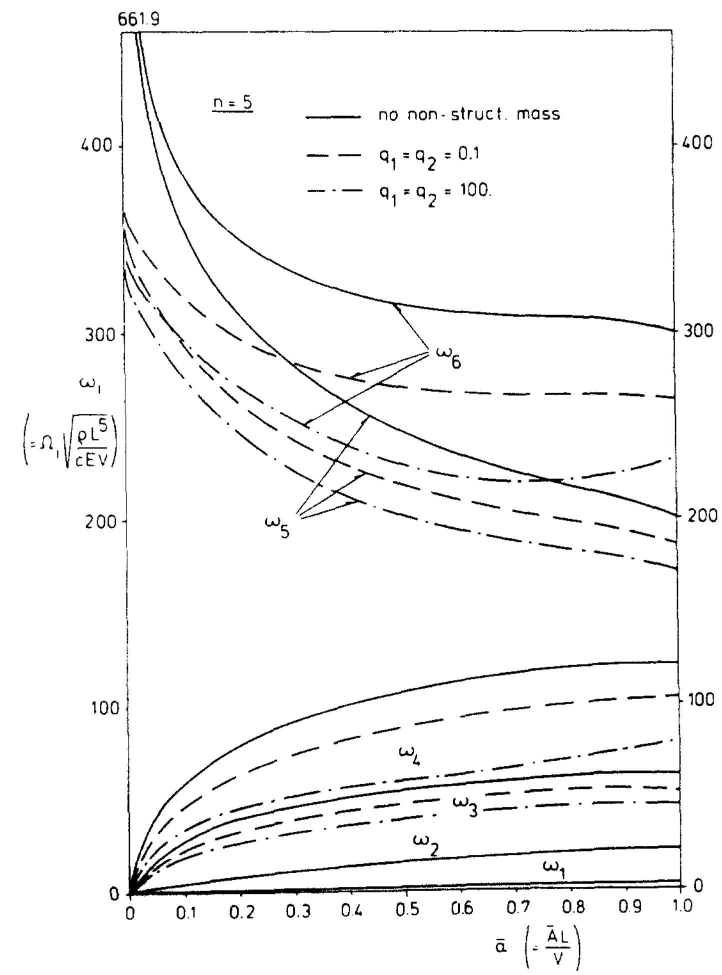

Fig. 2. Eigenfrequencies of optimal $\omega_5 - \omega_4$ cantilevers vs. minimum cross-sectional area constraint $\bar{\alpha}$.

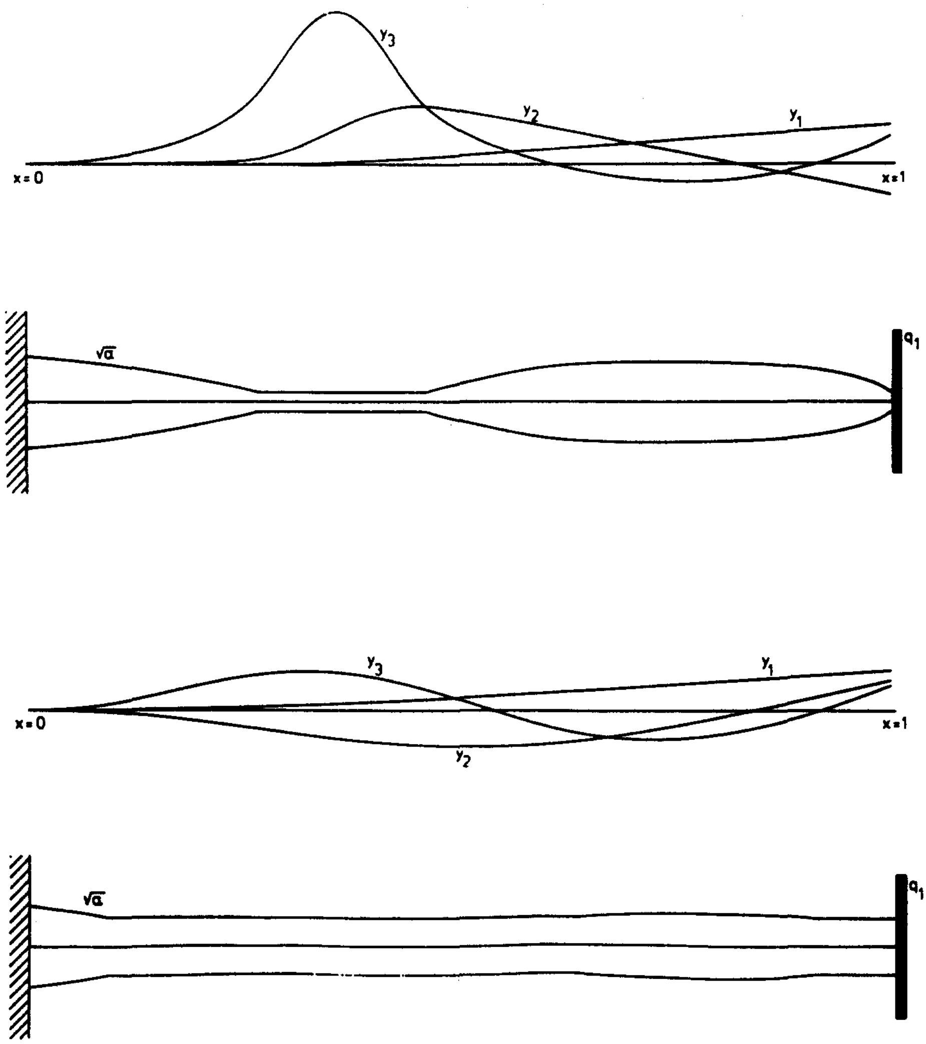

Fig. 3. Optimal $\omega_3 - \omega_2$ cantilevers with tip mass $q_1 = Q_1/\rho V = 0.1$. The solution above has $\bar{\alpha} = \bar{A}L/V = 0.1$, and has $\omega_3 = \Omega_3(\rho L^5/cEV)^{1/2} = 92.4$, $\omega_2 = 10.4$, and $(\omega_3 - \omega_2)/(\omega_3^u - \omega_2^u) = \Delta\omega/\Delta\omega^u = 2.27$. The solution below has $\bar{\alpha} = 0.9$, $\omega_3 = 60.0$, $\omega_2 = 20.4$, $\Delta\omega/\Delta\omega^u = 1.09$.

The limiting case $\bar{\alpha} = 1$ corresponds to fully constrained, i.e. uniform design with $\alpha(x) = 1$. On the other hand, the solution for the limiting case with $\bar{\alpha} = 0$ is geometrically unconstrained and corresponds to the solution of the problem discussed in Ref.[8], as explained in the Introduction.

In Figs. 1 and 2 the non-dimensional eigenfrequencies are plotted against the value of the minimum allowable cross-sectional area $\bar{\alpha}$. In Figs. 3 and 4 we present the optimal shape of

Table 1. Percentage improvement in the case $n = 3$ of Frequency difference of Optimal Beam compared with that of Beam with maximum value of ω_n

	q			
$\bar{\alpha}$	0.	0.1	1.	100.
0	0%	0%	0%	0%
0.2	1.3	0.94	0.95	1.0
0.4	1.6	1.5	1.5	1.6
0.6	1.4	1.3	1.3	1.5
0.8	0.7	0.5	0.5	0.6
1.0	0	0	0	0

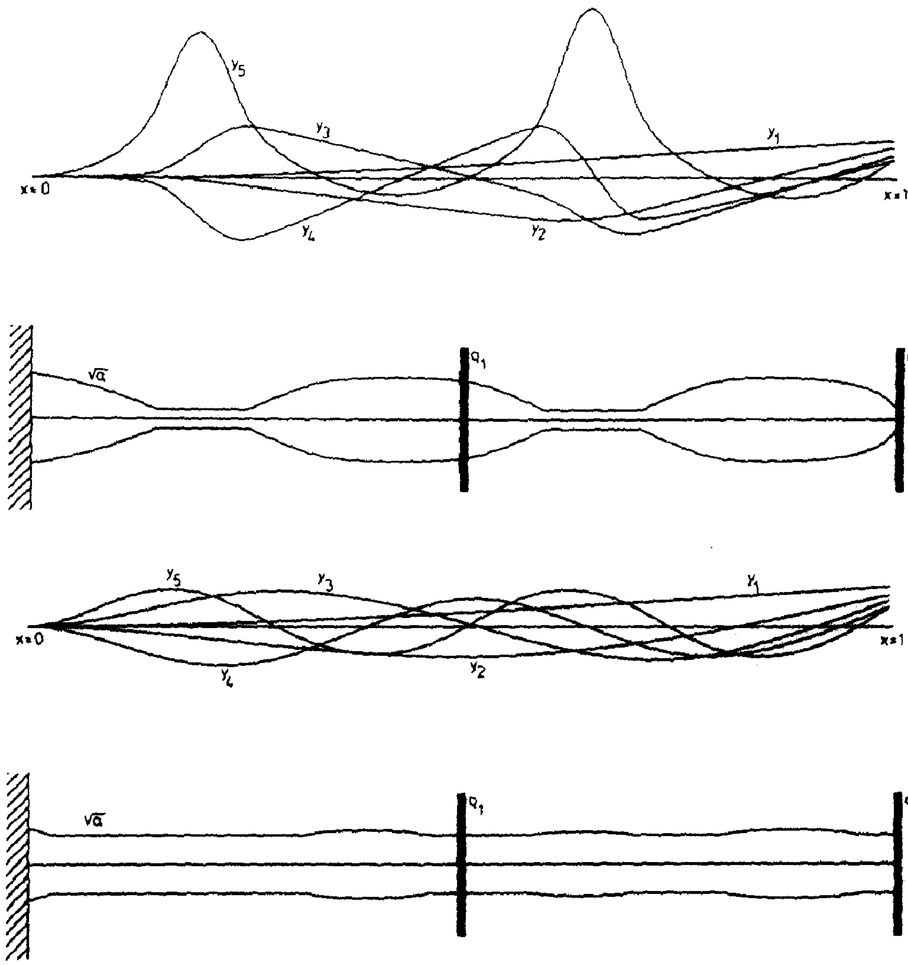

Fig. 4. Optimal $\omega_5 - \omega_4$ cantilevers with tip mass $q_1 = q_2 = 0.1$ at $x = \frac{1}{2}$ and $x = 1$ respectively. The solution above has $\bar{\alpha} = \bar{A}L/V = 0.1$, $\omega_5 = 296$, $\omega_4 = 48$ and $\Delta\omega/\Delta\omega^* = (\omega_5 - \omega_4)/(\omega_5^* - \omega_4^*) = 3.03$. The solution below has $\bar{\alpha} = 0.9$, $\omega_5 = 192$, $\omega_4 = 102$, $\Delta\omega/\Delta\omega^* = 1.11$.

some of the optimal beams for selected values of $\bar{\alpha}$. It should be noted that the linear dimensions of the cross-sections are proportional to $\sqrt{\alpha}$.

In order to assess the possible improvement brought about by the optimization, it is interesting to compare the optimum results with those of other designs. This is done in Figs. 5 and 6, and Tables 1 and 2. Figures 5 and 6 show, for $n = 3$, 5 respectively, the frequency separation of the optimal beam compared with that of a uniform reference cantilever of the same volume, length, and non-structural mass. The results are plotted against $\bar{\alpha}$. Tables 1 and 2 show the optimal frequency difference obtained here compared to that obtained in the simpler problem of maximizing ω_n.

Table 2. Percentage improvement in the case $n = 5$ of Frequency difference of Optimal Beam compared with that of Beam with maximum value of ω_n

	q_1, q_2			
$\bar{\alpha}$	0.	0.1	1.	100.
0	0%	0%	0%	0%
0.2	4.0	3.2	3.2	3.1
0.4	5.4	4.5	5.6	5.4
0.6	6.7	5.0	6.9	7.3
0.8	5.5	4.0	6.2	6.3
1.0	0	0	0	0

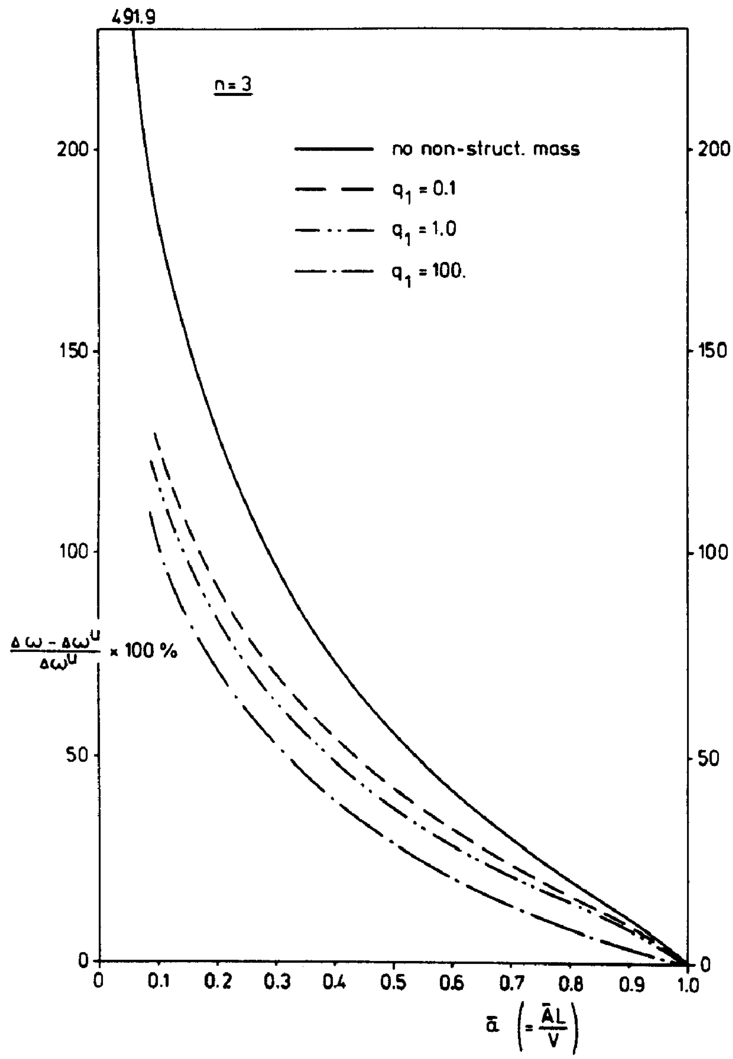

Fig. 5. Percentage improvement in frequency difference for $n = 3$. The solid line corresponds to optimal beams without non-structural mass and the other curves for beams with a single mass $q_1 = Q_1/\rho V$ at the tip.

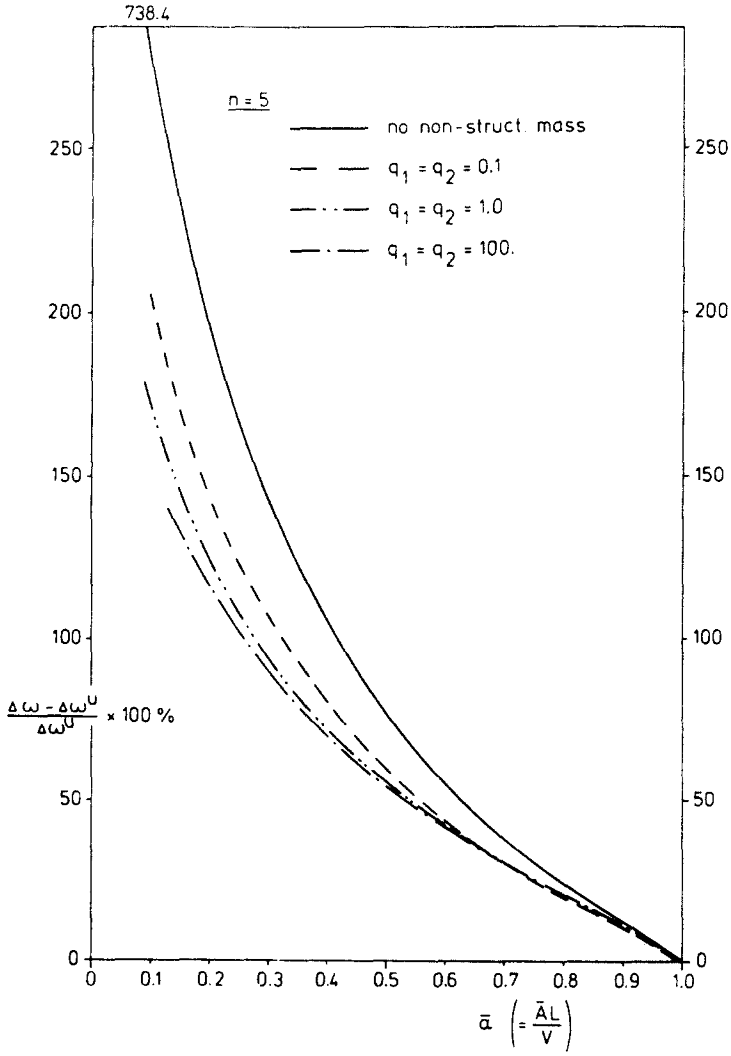

Fig. 6. Percentage improvement in frequency difference for $n = 5$. The solid line corresponds to optimal beams without non-structural mass and the other curves for beams with added masses $q_1 = q_2$ at $x = \frac{1}{2}$ and $x = 1$, respectively.

6. CONCLUSION

This paper demonstrates the maximization of the difference between adjacent higher natural frequencies of given order for transversely vibrating beams and rotating shafts. In comparison with a uniform beam or shaft, the frequency difference is substantially increased, particularly for small minimum cross-sectional area constraint values, $\bar{\alpha}$. The comparison with the beam or shaft having maximum value of ω_n (see Refs.[8, 9]) shows small, but consistent increases in frequency difference for the optimal design. However, it may be seen that neither the optimal shapes nor the natural frequencies differ greatly from those of the beam or shaft having maximum ω_n. Generally the value of ω_n for the present problem is marginally less than that obtained when ω_n alone is maximized, while the value of ω_{n-1} is slightly depressed compared to the simpler problem.

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